Inter (Part-II) 2018

Mathematics	Group-II	PAPER: II
Time: 2.30 Hours	(SUBJECTIVE TYPE)	Marks: 80

SECTION-I

2. Write short answers to any EIGHT (8) questions: (16)

(i) Prove that $\cosh^2 x + \sinh^2 x = \cosh 2x$.

Ans L.H.S $= \cosh^2 x + \sinh^2 x$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + e^{-2x} + 2e^x e^{-x}}{4} + \frac{e^{2x} + e^{-2x} - 2e^x e^{-x}}{4}$$

$$= \frac{e^{2x} + e^{-2x} + 2e^x e^{-x} + e^{2x} + e^{-2x} - 2e^x e^{-x}}{4}$$

$$= \frac{2e^{2x} + 2e^{-2x}}{4} = \frac{{}^1 2(e^{2x} + e^{-2x})}{4_2}$$

$$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x$$

Hence Proved.

L.H.S = R.H.S.

(ii) Determine whether function $f(x) = \dfrac{x^3 - x}{x^2 + 1}$ is even or odd.

Ans Let $f(x) = f(-x)$

So, $f(-x) = \dfrac{(-x)^3 - (-x)}{(-x)^2 + 1}$

$$= \frac{-x^3 + x}{x^2 + 1} = -\left[\frac{(x^3 - x)}{x^2 + 1}\right]$$

$f(-x) = -f(x)$

So $f(x)$ is an odd function.

(iii) Evaluate $\lim\limits_{x \to 0} \dfrac{\sec x - \cos x}{x}$.

Ans

$$\lim_{x \to 0} \frac{\sec x - \cos x}{x}$$

$$= \lim_{x \to 0} \left(\frac{\frac{1}{\cos x} - \cos x}{x}\right) \qquad \left(\because \sec x = \frac{1}{\cos x}\right)$$

$$= \lim_{x\to0} \frac{1-\cos^2 x}{x\cos x}$$

$$= \lim_{x\to0} \frac{\sin^2 x}{x\cos x}$$

$$= \lim_{x\to0} \frac{\sin^2 x}{x\cos x} \times \frac{x}{x}$$

$$= \lim_{x\to0} \left(\frac{\sin^2 x}{x^2} \times \frac{x}{\cos x}\right)$$

$$= \lim_{x\to0} \frac{\sin^2 x}{x^2} \times \lim_{x\to0} \frac{x}{\cos x}$$

$$(1)^2 \times 0 = 0$$

(iv) Find $\frac{dy}{dx}$ if $y = \frac{a+x}{a-x}$.

Ans As $y = \frac{a+x}{a-x}$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{a+x}{a-x}\right]$$

$$= \frac{(a-x)\frac{d}{dx}(a+x) - (a+x)\frac{d}{dx}(a-x)}{(a-x)^2}$$

$$= \frac{(a-x)(1)-(a+x)(-1)}{(a-x)^2}$$

$$= \frac{a-x+a+x}{(a-x)^2} = \frac{2a}{(a-x)^2}$$

(v) Find $\frac{dy}{dx}$ if $x^2 - 4xy - 5y = 0$.

Ans $x^2 - 4xy - 5y = 0$

$$\frac{d}{dx}(x^2 - 4xy - 5y) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(4xy) - \frac{d}{dx}(5y) = 0$$

$$\frac{d}{dx}(x^2) - 4\frac{d}{dx}(xy) - 5\frac{d}{dx}(y) = 0$$

$$2x - 4\left[x\cdot\frac{dy}{dx} + y\cdot 1\right] - 5\frac{dy}{dx} = 0$$

$$2x - 4x\frac{dy}{dx} - 4y - 5\frac{dy}{dx} = 0$$

$$4x\frac{dy}{dx}+5\frac{dy}{dx}=2x-4y$$

$$\frac{dy}{dx}(4x+5)=2(x-2y)$$

$$\frac{dy}{dx}=\frac{2(x-2y)}{4x+5}$$

(vi) Differentiate $x^2-\dfrac{1}{x^2}$ w.r.t x^4.

Ans Let $y=x^2-\dfrac{1}{x^2}$; $u=x^4$

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^2-\frac{1}{x^2}\right)$$

$$=\frac{d}{dx}(x^2)-\frac{d}{dx}\left(\frac{1}{x^2}\right)$$

$$=2x-\left[\frac{x^2(0)-1(2x)}{x^4}\right]$$

$$=2x-\frac{0-2x}{x^4}$$

$$=2x+\frac{2x}{x^4}$$

$$=2x+\frac{2}{x^3}$$

$$=\frac{2x^4+2}{x^3}$$

$$\frac{dy}{dx}=\frac{2(x^4+1)}{x^3}$$

As $u=x^4$

$$\frac{du}{dx}=4x^3$$

$$\frac{dy}{du}=\frac{dy}{dx}\times\frac{dx}{du}$$

$$=\frac{2(x^4+1)}{4x^3\,.\,x^3}=\frac{x^4+1}{2x^6}$$

(vii) Differentiate $\sin^{-1}\sqrt{1-x^2}$ w.r.t 'x'.

Ans Let $y=\sin^{-1}\sqrt{1-x^2}$

$$\frac{dy}{dx}=\frac{d}{dx}(\sin^{-1}\sqrt{1-x^2})$$

$$= \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{d}{dx}\sqrt{1-x^2}$$

$$= \frac{1}{\sqrt{1-1+x^2}}\left[\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right]$$

$$= \frac{1}{\sqrt{x^2}} \cdot \frac{-x}{\sqrt{1-x^2}} = \frac{1}{x} \cdot \frac{-x}{\sqrt{1-x^2}} = \frac{-1}{\sqrt{1-x^2}}$$

(viii) Find $\frac{dy}{dx}$ if $y = ln\left(x + \sqrt{x^2+1}\right)$.

Ans

$$\frac{dy}{dx} = \frac{d}{dx}\left[ln(x + \sqrt{x^2+1})\right]$$

$$= \frac{1}{x+\sqrt{x^2+1}} \frac{d}{dx}\left(x + \sqrt{x^2+1}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}} \cdot \left[1 + \frac{1}{2}(x^2+1)^{-1/2}(2x)\right]$$

$$= \frac{1}{x+\sqrt{x^2+1}} \cdot \left[1 + \frac{1x}{\sqrt{x^2+1}}\right] = \frac{1}{\sqrt{x^2+1}}$$

(ix) Find $\frac{dy}{dx}$ if $y = e^{-2x} \sin 2x$.

Ans

$$\frac{dy}{dx} = \frac{d}{dx}(e^{-2x} \sin 2x)$$

$$= e^{-2x}(\sin 2x)' + \sin 2x\,(e^{-2x})'$$

$$= e^{-2x} \cdot \cos 2x\,(2) + \sin 2x \cdot e^{-2x}(-2)$$

$$= 2e^{-2x}\cos 2x - 2e^{-2x}\sin 2x$$

$$= 2e^{-2x}(\cos 2x - \sin 2x)$$

(x) Find $\frac{d^2y}{dx^2}$ if $y^3 + 3ax^2 + x^3 = 0$.

Ans

$$\frac{d}{dx}(y^3 + 3ax^2 + x^3) = \frac{d}{dx}(0)$$

$$3y^2\frac{dy}{dx} + 3(a(2x) + x^2(0)) + 3x^2 = 0$$

$$3y^2\frac{dy}{dx} + 3(2ax) + 3x^2 = 0$$

$$3y^2\frac{dy}{dx} + 6ax + 3x^2 = 0$$

$$3y^2\frac{dy}{dx} = -6ax - 3x^2$$

$$\frac{dy}{dx} = \frac{-3(2ax + x^2)}{3y^2}$$

$$\frac{dy}{dx} = \frac{-(2ax + x^2)}{y^2}$$

Differentiate Again.

$$\frac{d^2y}{dx^2} = -\left[\frac{y^2(2ax + x^2)' - (2ax + x^2)(y^2)'}{y^4}\right]$$

$$= \frac{-\left[y^2(2a + 2x) - (2ax + x^2)\,2y\,\frac{dy}{dx}\right]}{y^4}$$

$$= \frac{-\left[2(a + x)y^2 - (2ax + x^2).2y\left(\frac{-2ax + x^2}{y^2}\right)\right]}{y^4}$$

$$= \frac{-2(a + x)y^2 - \frac{2(2ax + x^2)(2ax + x^2)}{y}}{y^4}$$

$$= \frac{-2\,[(a + x)y^3 + (2ax + x^2)^2]}{y^4\,.\,y}$$

Put $y^3 = -3ax^2 - x^3$

$$= \frac{-2[(a + x)(-3ax^2 - x^3) + 4x^2a^2 + 4ax^3 + x^4]}{y^5}$$

$$= \frac{-2[(a + x)x^2(-3a - x) + x^2(4a^2 + 4ax + x^2)]}{y^5}$$

$$= \frac{-2x^2\,[-(a + x)(3a + x) + 4a^2 + 4ax + x^2]}{y^5}$$

$$= \frac{-2x^2\,[-(3a^2 + 4ax + x^2) + 4a^2 + x^2 + 4ax}{y^5}$$

$$= \frac{-2x^2(a^2)}{y^5} = \frac{-2a^2x^2}{y^5}$$

(xi) Find y_2 if $y = \cos^3 x$.

Ans $\frac{dy}{dx} = \frac{d}{dx}(\cos^3 x)$

$= 3\cos^2 x\,(-\sin x)$

$= -3\cos^2 x \sin x$

$= -3(1 - \sin^2 x)\sin x$

$= -3\sin x + 3\sin^3 x$

Differentiate again

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-3\sin x + 3\sin^3 x)$$

$= -3\cos x + 9\sin^2 x\cos x$

$= -3\cos x + 9(1 - \cos^2 x)\cos x$

$= -3\cos x + 9\cos x - 9\cos^3 x$

$= 6\cos x - 9\cos^3 x$

(xii) **Find $\frac{dy}{dx}$ if $y = \ln\left(\frac{x^2 - 1}{x^2 + 1}\right)^{1/2}$.**

Ans Let $u = \frac{x^2 - 1}{x^2 + 1}$

$y = \ln\sqrt{u}$

$= \frac{1}{2}\ln u$

$$\frac{dy}{du} = \frac{1}{2}\frac{1}{u}$$

$$= \frac{1}{2} \cdot \frac{1}{\frac{x^2 - 1}{x^2 + 1}}$$

$$\frac{dy}{du} = \frac{x^2 + 1}{2(x^2 - 1)}$$

As $u = \frac{x^2 - 1}{x^2 + 1}$

Differentiate w.r.t 'x'

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{x^2 - 1}{x^2 + 1}\right)$$

$$= \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2}$$

$$= \frac{2x(x^2 + 1 - x^2 + 1)}{(x^2 + 1)^2}$$

$$= \frac{2x(2)}{(x^2 + 1)^2}$$

$$= \frac{4x}{(x^2 + 1)^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{x^2 + 1}{2(x^2 - 1)} \times \frac{4x}{(x^2 + 1)^2}$$

$$= \frac{2x(x^2 + 1)}{(x^2 - 1)(x^2 + 1)^2}$$

$$= \frac{2x}{(x^2 - 1)(x^2 + 1)} = \frac{2x}{x^4 - 1}$$

3. Write short answers to any EIGHT (8) questions: (16)

(i) **Find δy and dy : $y = \sqrt{x}$, when x changes from 4 to 4.41.**

Ans

$$y = \sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{2}(x)^{-1/2} \quad . (1)$$

$$dy = \frac{1}{2\sqrt{x}} dx$$

As x changes from 4 to 4.41

So, $x = 4$ and $\delta x = dx = 4.41 - 4 = 0.41$

$$dy = \frac{1}{2\sqrt{4}}(.41)$$

$$= \frac{1}{2\sqrt{(2)^2}}(.41)$$

$$= \frac{1}{4}(.41) = 0.1025$$

For change δx in x

$$y + \delta y = \sqrt{x + \delta x}$$

$$\delta y = \sqrt{x + \delta x} - y = \sqrt{x + \delta x} - \sqrt{x}$$

If $x = 4$ & $\delta x = .41$

$$\delta y = \sqrt{4 + .41} - \sqrt{4} = \sqrt{4.41} - \sqrt{4}$$

$$= 2.1 - 2 = 0.1$$

(ii) **Evaluate $\int \frac{e^{2x} + e^x}{e^x} dx$.**

Ans

$$I = \int \left(\frac{e^{2x}}{e^x} + \frac{e^x}{e^x}\right) dx$$

$$= \int (e^x + 1)\, dx$$

$$= \int e^x\, dx + \int 1\, dx$$

$$I = e^x + x + c$$

(iii) Evaluate $\int (a - 2x)^{3/2}\, dx$.

Ans $\int (a - 2x)^{3/2} \times \left(\frac{-2}{-2}\right) dx$

$$= \frac{-1}{2} \int (a - 2x)^{3/2} (-2)\, dx$$

$$= \frac{-1}{2} \left[\frac{(a - 2x)^{3/2+1}}{\frac{3}{2} + 1} \right] + c$$

$$= \frac{-1}{2} \frac{(a - 2x)^{5/2}}{\frac{5}{2}} + c$$

$$= \frac{-1}{2} \cdot \frac{2}{5} (a - 2x)^{5/2} + c$$

$$= \frac{-1}{5} (a - 2x)^{5/2} + c$$

(iv) Evaluate $\int \frac{x + b}{(x^2 + 2bx + c)^{1/2}}\, dx$.

Ans $= \int \frac{x + b}{(x^2 + 2bx + c)^{1/2}}\, dx$

$$= \int (x^2 + 2bx + c)^{-1/2} (x + b)\, dx$$

$$= \frac{1}{2} \int (x^2 + 2bx + c)^{-1/2} . (2x + 2b)\, dx$$

$$= \frac{1}{2} \left[\frac{(x^2 + 2bx + c)^{-1/2+1}}{-\frac{1}{2} + 1} \right] + c_1$$

$$= \frac{1}{2} \left[\frac{(x^2 + 2bx + c)^{1/2}}{\frac{1}{2}} \right] + c_1$$

$$= \frac{1}{2} . 2 (x^2 + 2bx + c)^{1/2} + c_1$$

$$= \sqrt{x^2 + 2bx + c} + c_1$$

(v) Evaluate $\int x\, e^x\, dx$.

Ans Integrate by parts

$$= x\, e^x - \int e^x (1)\, dx$$

$= x e^x - e^x + c$

(vi) Evaluate $\int e^x \left(\frac{1}{x} + \ln x\right) dx$.

Ans We can also write as

$$\int e^x \left(\ln x + \frac{1}{x}\right) dx$$

If $f(x) = \ln x$

Then $f'(x) = \frac{1}{x}$

So $\int e^x (f(x) + f(x')) dx$

$= e^x f(x) + c$

$= e^x \ln x + c$

(vii) Evaluate $\int_{-1}^{3} (x^3 + 3x^2) dx$.

Ans $\int_{-1}^{3} (x^3 + 3x^2) dx$

$$= \int_{-1}^{3} x^3 dx + 3 \int_{-1}^{3} x^2 dx$$

$$= \left|\frac{x^4}{4}\right|_{-1}^{3} + 3.\frac{1}{3} |x^3|_{-1}^{3}$$

$$= \frac{1}{4}[(3)^4 - (1)^4] + [3^3 - (-1)^3]$$

$$= \frac{1}{4}[81 - 1] + [27 + 1] = 20 + 28 = 48$$

(viii) Evaluate $\int_{0}^{\pi/3} \cos^2\theta \sin\theta \, d\theta$.

Ans $$I = \int_{0}^{\pi/3} \cos^2\theta \sin\theta \, d\theta$$

If $f(\theta) = \cos\theta$

$f'(\theta) = -\sin\theta$

$$= -\int_{0}^{\pi/3} \cos^2\theta (-\sin\theta) \, d\theta$$

$$= -\left[\frac{\cos^3\theta}{3}\right]_{0}^{\pi/3} + c$$

$$= -\frac{1}{3}\left[\cos^3\frac{\pi}{3} - \cos^3 0\right] + c$$

$$= -\frac{1}{3}\left[\left(\frac{1}{2}\right)^3 - (1)^3\right] + c$$

$$= -\frac{1}{3}\left(\frac{1}{8} - 1\right) + c$$

$$= -\frac{1}{3}\left(\frac{7}{8}\right) \Rightarrow \frac{-7}{24}$$

(ix) Find the area between the x-axis and the curve $y = 4x - x^2$ from $x = 0$ to $x = 4$.

Ans Given equation of curve is $y = 4x - x^2$

To find x-intercept

Put $y = 0$

$$4x - x^2 = 0$$

$$x(4 - x) = 0$$

$$\boxed{x = 0, 4}$$

So given curve cuts x-axis at points (0, 0), (4, 0).

As $y = 4x - x^2 \geq 0$ for $0 \leq x \leq 4$

Thus the area bounded by given curve is above x-axis.

If A be the required area, then

$$A = \int_0^4 y\,dx$$

$$A = \int_0^4 (4x - x^2)\,dx$$

$$= \left[4\frac{x^2}{2} - \frac{x^3}{3}\right]_0^4$$

$$= \left[2x^2 - \frac{x^3}{3}\right]_0^4$$

$$= \left[2(4)^2 - \frac{(4)^3}{3}\right] - \left[2(0)^2 - \frac{(0)^3}{3}\right]$$

$$= \left[2(16) - \frac{64}{3}\right] - [0 - 0]$$

$$= 32 - \frac{64}{3}$$

$$= \frac{96 - 64}{3}$$

$$\boxed{A = \frac{32}{3} \text{ square units}}$$

(x) Define differential equation.

Ans An equation having at least one derivative of a dependent variable w.r.t an independent variable.

(xi) Solve $\frac{dy}{dx} = \frac{y^2 + 1}{e^{-x}}$.

Ans By separating the variable, we have

$$\frac{dy}{y^2 + 1} = \frac{1}{e^{-x}} dx$$

$$\int \frac{1}{y^2 + 1} dy = \int e^x dx$$

$$\tan^{-1} y = e^x + c$$

$$y = \tan (e^x + c)$$

(xii) Solve $\frac{dy}{dx} = 2x$.

Ans By separating the variables, we have

$$dy = 2x \, dx$$

Integrate on both sides,

$$\int dy = \int 2x \, dx$$

$$y^2 = \frac{2x^2}{2} + c$$

$$y = x^2 + c$$

4. Write short answers to any NINE (9) questions: (18)

(i) Write down equation of straight line with x-intercept (2, 0) and y-intercept (0, –4).

Ans Equation of a line whose none-zero x and y intercepts is:

$$\frac{x}{a} + \frac{y}{b} = 1$$

In x-intercept $a = 2, b = 0$

In y-intercept $a = 0, b = -4$

By putting values in the formula

$$\frac{x}{2}+\frac{y}{-4}=1$$

$$\frac{2x-y}{4}=1$$

$$2x-y=4$$

$$2x-y-4=0 \qquad \text{(Required equation)}$$

(ii) Find an equation of a line bisecting 2^{nd} and 4^{th} quadrants.

Ans The line passes through (0, 0) and having slope-1, so its equation is:

$$y=-x$$

(iii) Find an equation of a line with x-intercept: –9 and slope: –4.

Ans Given points are (–9, 0) and m = –4

The equation of required line is

$$y-y_1=m(x-x_1)$$

$$y-0=-4(x-(-9))$$

$$y=-4(x+9)$$

$$y=-4x-36$$

$$y+4x+36=0$$

(iv) Prove that if the lines are perpendicular, then product of their slopes = –1.

Ans As the lines are perpendicular to each other. So,

Let,

$$\text{Inclimation of } l_1=\alpha$$

$$\text{Inclimation of } l_2=\alpha+90$$

$$m_1=\tan\alpha$$

$$m_2=\tan(90+\alpha)=-\cot\alpha$$

$$\text{Product of } m_1, m_2=m_1\times m_2$$

$$=\tan\alpha\times(-\cot\alpha)$$

$$=\tan\alpha\times\frac{-1}{\tan\alpha}$$

Product of their slopes = –1.

(v) Find the measure of angle between the lines represented by $x^2-xy-6y^2=0$.

Ans Here

$$a = 1,\ h = \frac{-1}{2},\ b = -6$$

If θ is measure of the angle between the given lines, then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$= \frac{2\sqrt{\left(\frac{-1}{2}\right)^2 - (1)(-6)}}{1-6} = \frac{2\sqrt{\frac{1}{4} + 6}}{-5} = \frac{2\sqrt{\frac{25}{4}}}{-5}$$

$= -1$

$= \theta$

$= 135^\circ$

Acute angle between the lines $= 180^\circ - \theta$

$= 180^\circ - 135^\circ$

$= 45^\circ$

(vi) Find focus and vertex of the parabola $y = 6x^2 - 1$.

Ans

$$y = 6x^2 - 1$$

$$\frac{y+1}{6} = x^2$$

Let: $x = X$ and $Y = y + 1$

So $\frac{1}{6}Y = X^2$ (i)

Comparing equation (i) with $X^2 = 4aY$ we have,

$$4a = \frac{1}{6}$$

$$a = \frac{1}{24}$$

Vertex of equation (i) is (0, 0)

$X = 0$ $\quad$ $Y = 0$

$x = 0$ $\quad$ $y + 1 = 0$

$y = -1$

vertex = (0, −1)

Focus of parabola = (0, a)

$$= \left(0, \frac{1}{24}\right)$$

$$X = 0; \qquad Y = \frac{1}{24}$$

$$x = 0;\ y + 1 = \frac{1}{24}$$

$$y = \frac{1}{24} - 1$$

$$= \frac{1 - 24}{24}$$

$$y = \frac{-23}{24}$$

$$\text{Focus} = \left(0, \frac{-23}{24}\right)$$

(vii) Find equation of latus rectum of parabola $y^2 = -8(x - 3)$.

Ans $y^2 = -8(x - 3)$

Let $y = Y$ and $X = x - 3$

$y^2 = -8X$

Compare this equation with $Y^2 = 4aX$

We have

$4a = -8$

$a = -2$

Equation of Latus rectum = $x - a = 0$

$x - 3 - (-2) = 0$

$x - 3 = -2$

$x = -2 + 3$

$\boxed{x = 1}$

(viii) Find an equation of an ellipse with foci (± 3, 0) and minor axis of length 10.

Ans Here foci $(\pm c, 0) = (\pm 3, 0)$

$c = ae = 3$

$2b = 10$

$\Rightarrow$ $b = 5$

As $c^2 = a^2 - b^2$

Put values of b and c,

$(3)^2 = a^2 - (5)^2$

$9 = a^2 - 25$

$9 + 25 = a^2$

$\Rightarrow$ $a^2 = 34$

$a = \sqrt{34}$

Since major axis is along x-axis. So the equation of ellipse is $\frac{x^2}{34}+\frac{y^2}{25}=1$.

(ix) Find the foci and length of the latus rectum of the ellipse $9x^2 + y^2 = 18$.

Ans $9x^2 + y^2 = 18$

Divide by 18, we have

$$\frac{x^2}{2}+\frac{y^2}{18}=1 \qquad \left[\because \frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \;;\; a>b\right]$$

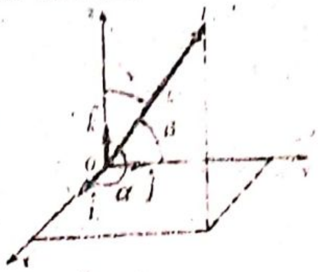
$$a^2 = 18,\ b^2 = 2$$
$$c^2 = a^2 - b^2$$
$$= 18 - 2$$
$$c^2 = 16$$
$$c = \pm 4$$
$$\text{Foci} = (0, \pm c)$$
$$\boxed{F = (0, \pm 4)}$$

$$\text{Length of Latus Rectum} = \frac{2a^2}{b}$$
$$= \frac{2(18)}{\sqrt{2}}$$
$$= \frac{36}{\sqrt{2}}$$

(x) Define direction angles and direction cosines of a vector.

Ans Let $r = \overrightarrow{OP} = x\overline{i} + y\overline{j} + z\overline{k}$ be a non-zero vector. Let α, β, γ denoted the angles formed between r and the unit coordinate vectors.

α, β, γ are called direction angles and cos α, cos β and cos γ are called direction cosines.



(xi) Find the projection of vector $\underline{a}$ along vector $\underline{b}$ and projection of vector $\underline{b}$ along $\underline{a}$ when $\underline{a} = \hat{i} - \hat{k}$, $b = \hat{j} + \hat{k}$.

Ans Projection of $\vec{a}$ along $\vec{b} = \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|}$

$$\underline{b} = \underline{j} + \underline{k}$$

$$|\underline{b}| = \sqrt{(1)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$\underline{a} \cdot \underline{b} = (\underline{i} - \underline{k}) \cdot (\underline{j} + \underline{k})$$

$$= (\underline{i} + 0\underline{j} - \underline{k}) \cdot (0\underline{i} + \underline{j} + \underline{k})$$

$$= 1 \times 0 + 0 \times 1 + (-1)(1)$$

$$= -1$$

Projection of $\vec{a}$ along $\vec{b} = \frac{-1}{\sqrt{2}}$.

(xii) Find a vector perpendicular to each of the vectors $\underline{a} = 2\hat{i} + \hat{j} + \hat{k}$ and $\underline{b} = 4\hat{i} + 2\hat{j} - \hat{k}$.

Ans Let $\underline{c}$ be the vector perpendicular to both $\underline{a}$ and $\underline{b}$.

$$\underline{c} = \underline{a} \times \underline{b}$$

$$\underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 1 \\ 4 & 2 & -1 \end{vmatrix}$$

$$= (-1-2)\,\underline{i} - (-2-4)\,\underline{j} + (4-4)\,\underline{k}$$

$$= -3\underline{i} + 6\underline{j} + 0\underline{k}$$

$$\underline{c} = -3\underline{i} + 6\underline{j}$$

(xiii) Convert $2x - 4y + 11 = 0$ into slope intercept form.

Ans

$$2x - 4y + 11 = 0$$

$$2x + 11 = 4y$$

$$\frac{x}{2} + \frac{11}{4} = y$$

SECTION-II

NOTE: Attempt any Three (3) questions.

Q.5.(a) Prove that $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$. **(5)**

Ans Put $a^x - 1 = y$

$$a^x = 1 + y$$

$$x = \log_a (1 + y)$$

When $x \to 0, y \to 0$

$$\lim_{x\to 0}\frac{a^x-1}{x}=\lim_{y\to 0}\frac{y}{\log_a(1+y)}$$

$$=\lim_{y\to 0}\frac{1}{\frac{1}{y}\log_a(1+y)}$$

$$=\lim_{y\to 0}\frac{1}{\log_a(1+y)^{1/y}}$$

$$=\frac{1}{\log_a e}$$

$$=\log_e a$$

(b) Prove that $y\frac{dy}{dx}+x=0$ if $x=\frac{1-t^2}{1+t^2}$, $y=\frac{2t}{1+t^2}$. (5)

Ans $x=\frac{1-t^2}{1+t^2}$ and $y=\frac{2t}{1+t^2}$

$$x^2+y^2=\left(\frac{1-t^2}{1+t^2}\right)^2+\left(\frac{2t}{1+t^2}\right)^2$$

$$=\frac{(1-t^2)^2+(2t)^2}{(1+t^2)^2}$$

$$=\frac{\cancel{1+t^4}-2t^2+4t^2}{\cancel{1+t^4}+2t^2}$$

$$=\frac{2t^2}{2t^2}$$

$x^2+y^2=1$

Differentiating w.r.t 'x'

$$2x+2y\frac{dy}{dx}=0$$

$$2\left(x+y\frac{dy}{dx}\right)=0$$

$$x+y\frac{dy}{dx}=0 \qquad \text{Hence proved.}$$

Q.6.(a) Show that $\int\frac{dx}{\sqrt{x^2-a^2}}=\ln\left(x+\sqrt{x^2-a^2}\right)+c$. (5)

Ans $\int\frac{dx}{\sqrt{x^2-a^2}}=\ln\left(x+\sqrt{x^2-a^2}\right)+c$

Put $x=a\sec\theta$

$dx = a \sec\theta \tan\theta \, d\theta$

$\therefore \quad \int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sec\theta \tan\theta \, d\theta}{\sqrt{a^2 \sec^2\theta - a^2}}$

$= \int \frac{a \sec\theta \tan\theta \, d\theta}{a\sqrt{\tan^2\theta}} = \int \sec\theta \, d\theta$

$= \ln(\sec\theta + \tan\theta) + c_1$

since $\sec\theta = \frac{x}{a} \quad \therefore \quad \tan\theta \sqrt{\sec^2\theta - 1} = \sqrt{\frac{x^2}{a^2} - 1} = \frac{\sqrt{x^2 - a^2}}{a^2}$

$\therefore \quad \int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left(\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right)$

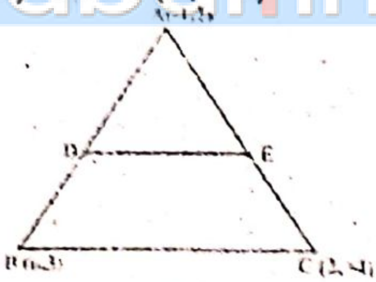
$= \ln(x + \sqrt{x^2 - a^2}) - \ln a + c_1$

Let, $\quad -\ln a + c_1 = c$

So $\quad \int \frac{dx}{\sqrt{x^2 - a^2}} = \ln(x + \sqrt{x^2 - a^2}) + c$

(b) **The points A(–1, 2), B(6, 3) and C(2, –4) are vertices of a triangle, then show that the line joining the mid-point "D" of $\overline{AB}$ and mid-point "E" of $\overline{AC}$ is parallel to $\overline{BC}$ and $\overline{DE} = \frac{1}{2}\overline{BC}$.** **(5)**

Ans A(–1, 2), B(6, 3) and C(2, –4)



Then coordinates of D $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$= \left(\frac{-1 + 6}{2}, \frac{2 + 3}{2}\right)$

$D = \left(\frac{5}{2}, \frac{5}{2}\right)$

Coordinates of E $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$$= \left(\frac{-1+2}{2}, \frac{2-4}{2}\right)$$

$$= \left(\frac{1}{2}, \frac{-2}{2}\right)$$

$$E = \left(\frac{1}{2}, -1\right)$$

$$\text{Slope of side } \overline{BC} = m_1 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-4-3}{2-6} = \frac{-7}{-4} = \frac{7}{4}$$

$$\text{Slope of side } \overline{DE} = m_2 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-1 - \frac{5}{2}}{\frac{1}{2} - \frac{5}{2}} = \frac{\frac{-2-5}{2}}{\frac{1-5}{2}} = \frac{\frac{-7}{2}}{\frac{-4}{2}}$$

$$= \frac{\frac{-7}{2}}{-2} = \frac{7}{4}$$

As $m_1 = m_2 = \frac{7}{4}$; So $\overline{DE} \parallel \overline{BC}$

$$|\overline{BC}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(2-6)^2 + (-4-3)^2}$$

$$= \sqrt{(-4)^2 + (-7)^2}$$

$$= \sqrt{16 + 49} = \sqrt{65}$$

$$|\overline{DE}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{\left(\frac{1}{2} - \frac{5}{2}\right)^2 + \left(-1 - \frac{5}{2}\right)^2}$$

$$= \sqrt{\left(\frac{-1-5}{2}\right)^2 + \left(\frac{-2-5}{2}\right)^2}$$

$$= \sqrt{\left(\frac{-4}{2}\right)^2 + \left(\frac{-7}{2}\right)^2}$$

$$= \sqrt{(-2)^2 + \left(\frac{-7}{2}\right)^2} = \sqrt{4 + \frac{49}{4}}$$

$$= \sqrt{\frac{16 + 49}{4}} = \sqrt{\frac{65}{4}} = \sqrt{\frac{65}{2}}$$

$$|\overline{DE}| = \frac{1}{2}\sqrt{65} = \frac{1}{2}|\overline{BC}| \quad \text{Hence proved.}$$

Q.7.(a) Evaluate $\int_0^{\pi/4} \cos^4 t \, dt$. (5)

Ans

$$= \frac{1}{4}\int_0^{\pi/4} 4\cos^4 t \, dt$$

$$= \frac{1}{4}\int_0^{\pi/4} (2\cos^2 t)^2 \, dt$$

$$= \frac{1}{4}\int_0^{\pi/4} (1 + \cos 2t)^2 \, dt$$

$$= \frac{1}{4}\int_0^{\pi/4} (1 + \cos^2 2t + 2\cos 2t) \, dt$$

$$= \frac{1}{4}\int_0^{\pi/4} \left(1 + \frac{1 + \cos 4t}{2} + 2\cos 2t\right) dt$$

$$= \frac{1}{4}\int_0^{\pi/4} \left(1 + \frac{1}{2} + \frac{1}{2}\cos 4t + 2\cos 2t\right) dt$$

$$= \frac{1}{4}\int_0^{\pi/4} \left(\frac{3}{2} + \frac{1}{2}\cos 4t + 2\cos 2t\right) dt$$

$$= \frac{1}{4}\left[\frac{3}{2}\int_0^{\pi/4} 1 \, dt + \frac{1}{2}\int_0^{\pi/4} \cos 4t \, dt + 2\int_0^{\pi/4} \cos 2t \, dt\right]$$

$$= \frac{1}{4}\left[\frac{3}{2}|t|_0^{\pi/4} + \frac{1}{2}\left|\frac{\sin 4t}{4}\right|_0^{\pi/4} + 2\left|\frac{\sin 2t}{2}\right|_0^{\pi/4}\right.$$

$$= \frac{1}{4}\left[\frac{3}{2}\left[\frac{\pi}{4} - 0\right] + \frac{1}{8}\left[\sin 4\frac{\pi}{4} - \sin 0\right] + \left(\sin 2\frac{\pi}{4} - \sin 0\right)\right]$$

$$= \frac{1}{4}\left[\frac{3\pi}{8} + \frac{1}{8}(\sin \pi - 0) + \left(\sin\frac{\pi}{2} - 0\right)\right]$$

$$= \frac{1}{4}\left[\frac{3\pi}{8} + \frac{1}{8}(0 - 0) + (1 - 0)\right]$$

$$= \frac{1}{4}\left[\frac{3\pi}{8} + 1\right] = \frac{1}{4}\left(\frac{3\pi + 8}{8}\right)$$

$$= \frac{3\pi + 8}{32}$$

(b) **Graph the feasible region of system of linear inequalities and find the corner points** **(5)**

$2x + 3y \le 18,\ x + 4y \le 12,\ 3x + y \le 12 \qquad x \ge 0,\ y \ge 0$

Ans

$$2x + 3y \le 18 \qquad (i)$$
$$x + 4y \le 12 \qquad (ii)$$
$$3x + y \le 12 \qquad (iii)$$

In equation (iii), put y = 0, x = 0

$$3x + 0 \le 12 \qquad 3(0) + y \le 12$$
$$3x \le 12 \qquad y \le 12$$
$$x \le \frac{12}{3} = 4$$

Corner point = (4, 0) (0, 12)

Put x = 0, y = 0 in equation (ii),

$$0 + 4y \le 12 \qquad x + 4(0) \le 12$$
$$4y \le 12 \qquad x \le 12$$
$$y \le \frac{12}{4}$$
$$y \le 3$$

Corner points (0, 3) (12, 0)

Compare equations (ii) and (iii),

Multiply eq. (ii) by '3' and then subtract

$$3x + 12y \le 36$$
$$\underline{3x \pm \quad y \le 12}$$
$$11y \le 24$$
$$y \le \frac{24}{11}$$

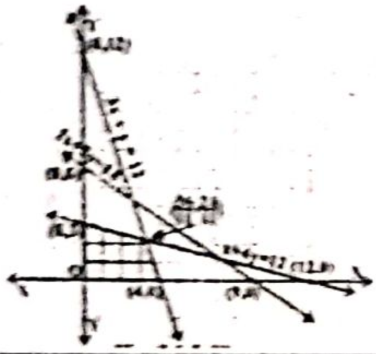
Put $y \le \frac{24}{11}$ in eq. (iii),

$$3x + \frac{24}{11} \le 12$$
$$3x \le 12 - \frac{24}{11}$$
$$3x \le \frac{132 - 24}{11}$$
$$x \le \frac{108}{3 \times 11}$$
$$x \le \frac{108}{33}$$

$$x \le \frac{36}{11}$$

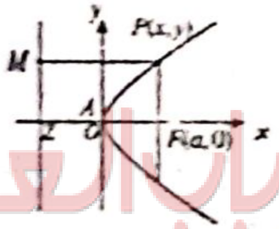
Corner points $\left(\frac{24}{11}, \frac{36}{11}\right)$

So corner points are (0, 0) (4, 0) (0, 3) $\left(\frac{24}{11}, \frac{36}{11}\right)$.



Q.8.(a) Find an equation of parabola having its focus at the origin and directrix parallel to y-axis. (5)

Ans Let focus f(a, θ) be the focus of parabola and x = –a the equation of directrix.



Also let P(x, y) be a point on the parabola and M be the point on directrix. Then

$$\frac{|\overline{PF}|}{|\overline{PM}|} = 1$$

$$|\overline{PF}| = |\overline{PM}|$$

$$M = x = -a$$

$$|PM| = x + a$$

$$|PF| = \sqrt{(x-a)^2 + (y-0)^2}$$

$$= \sqrt{(x-a)^2 + y^2}$$

As $|PF| = |PM|$

$$\sqrt{(x-a)^2 + y^2} = x + a$$

Taking square on both sides,

$$\left(\sqrt{(x-a)^2 + y^2}\right)^2 = (x+a)^2$$

$$(x-a)^2 + y^2 = (x+a)^2$$

$$y^2 = (x + a)^2 - (x - a)^2$$
$$= (x^2 + a^2 + 2ax) - (x^2 + a^2 - 2ax)$$
$$= x^2 + a^2 + 2ax - x^2 - a^2 + 2ax$$
$$= 4ax$$
$$y^2 = 4ax$$

which is the standard equation of parabola.

(b) Prove that the line segment joining the mid-points of two sides of a triangle is parallel to the third side and half as long. (5)

Ans We suppose that $\underline{a}$, $\underline{b}$, $\underline{c}$ are position vectors of triangle having vertices A, B and C respectively. We further suppose that P and Q be the mid-points of side AB and AC.

As P.V of $P = \frac{a + b}{2}$

and P.V of Q $= \frac{a + c}{2}$

$\overrightarrow{PQ}$ = P.V of Q – P.V of P

$$= \frac{a + c}{2} - \frac{a + b}{2} = \frac{c - b}{2} \qquad \text{(i)}$$

Now $\overrightarrow{BC}$ = P.V of C – P.V of B

$$= c - b \qquad \text{(ii)}$$

$$\frac{\overrightarrow{BC}}{2} = \frac{c - b}{2}$$

Therefore, from (i) and (ii),

$$\overrightarrow{PQ} = \frac{\overrightarrow{BC}}{2} = \frac{1}{2}\overrightarrow{BC} \qquad \text{(iii)}$$

Therefore, vector $\overrightarrow{PQ} \parallel \overrightarrow{BC}$ and it is clear from (iii) length of $\overrightarrow{PQ}$ is half as long as $\overrightarrow{BC}$.

Q.9.(a) Find the centre, foci, eccentricity, vertices and equations of directrices of $\frac{y^2}{4} - x^2 = 1$. (5)

Ans For Answer see Paper 2018 (Group-I), Q.9.(a).

(b) **Find the value of α, in the coplanar vectors $\alpha\hat{i} + \hat{j}$, $\hat{i} + \hat{j} + 3\hat{k}$, $2\hat{i} + \hat{j} - 2\hat{k}$.** **(5)**

Ans: Let $\underline{u} = \alpha\underline{i} + \underline{j}$

$\underline{v} = \underline{i} + \underline{j} + 3\underline{k}$.

$\underline{w} = 2\underline{i} + \underline{j} - 2\underline{k}$.

Given $\underline{u} \cdot (\underline{v} \times \underline{w}) = 0$

$$\begin{vmatrix} \alpha & 1 & 0 \\ 1 & 1 & 3 \\ 2 & 1 & -2 \end{vmatrix} = 0$$

$\alpha(-2 - 3) - 1(-2 - 6) + 0\,(1 - 2) = 0$

$-5\alpha + 8 = 0$

$-5\alpha = -8$

$\alpha = \frac{-8}{-5}$

$$\boxed{\alpha = \frac{8}{5}}$$